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# The unidirectional motion of two heat-conducting liquids in a flat channel

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**Abstract.** The unidirectional motion of two viscous incompressible liquids in a flat channel is studied. Liquids contact on a flat interface. External boundaries are fixed solid walls, on which the non-stationary temperature gradients are given. The motion is induced by a joint action of thermogravitational and thermocapillary forces and given total non - stationary fluid flow rate in layers. The corresponding initial boundary value problem is conjugate and inverse because the pressure gradients along axes channel have to be determined together with the velocity and temperature field. For this problem the exact stationary solution is found and a priori estimates of non - stationary solutions are obtained. In Laplace images the solution of the non - stationary problem is found in quadratures. It is proved, that the solution converges to a steady regime with time, if the temperature on the walls and the fluid flow rate are stabilized. The numerical calculations for specific liquid media good agree with the theoretical results.

## 1. Statement of the problem

We consider the system of two incompressible immiscible liquids with the interface  $y = 0$ . The parameters of liquids moving in strips  $-h_1 < y < 0$  and  $0 < y < h_2$ ,  $-\infty < x < \infty$  are denoted by the indices “1” and “2”, respectively;  $\rho_j$ ,  $\nu_j$ ,  $\chi_j$ ,  $c_{pj}$ ,  $\beta_j$ ,  $k_j$  are the densities, kinematic viscosities, coefficients of thermal diffusivity, specific heat of liquids, volumetric expansion and thermal conductivity, respectively. Further, it is assumed that these parameters are positive constants. The motion of liquids is described by the Oberbeck - Boussinesq equation system. Following the works [1, 2], we set

$$u_j = w_j(y, t), \quad \theta_j = -a_j(y, t)x + T_j(y, t), \quad p_j = -b_j(y, t)x + P_j(y, t), \quad j = 1, 2. \quad (1)$$

Substitution (1) in the Oberbeck - Boussinesq equation system leads to equations

$$\begin{aligned} a_{jt} &= \chi_j a_{jyy}, & b_{jy} &= \rho_j g \beta_j a_j, & w_{jt} &= \nu_j w_{jyy} + \frac{1}{\rho_j} b_j, \\ T_{jt} &= \chi_j T_{jyy} + a_j w_j, & P_{jy} &= \rho_j g \beta_j T_j, \end{aligned} \quad (2)$$

where  $u_j$  is the projection of the velocity vector on the axis  $x$ ,  $p_j$  is the deviation of pressure from hydrostatic one,  $g = \text{const}$  is the acceleration of gravity force,  $\theta_j$  is the temperature. Further, a problem for determining the velocity field (i. e. for  $w_j$  and  $a_j$ ) will be considered only. The functions  $T_j$  is the solution of the conjugate problem that is analogous to the problem for  $a_j$ .



The functions  $b_j$  and  $P_j$  are found from the second and fourth equations (2) with known  $a_j$  and  $T_j$ .

The conjugate boundary value problem for the functions  $a_j(y, t)$  has the form

$$a_{jt} = \chi_j a_{jyy}, \quad (3)$$

$$a_j(y, 0) = a_{j0}(y), \quad (4)$$

$$a_1(-h_1, t) = A_1(t), \quad a_2(h_2, t) = A_2(t), \quad (5)$$

$$a_1(0, t) = a_2(0, t), \quad k_1 a_{1y}(0, t) = k_2 a_{2y}(0, t) \quad (6)$$

with given functions  $a_{j0}(y)$ ,  $A_j(t)$ ,  $j = 1, 2$ , which for the smooth solutions must satisfy the matching conditions  $a_{10}(-h_1) = A_1(0)$ ,  $a_{20}(h_2) = A_2(0)$ ,  $a_{10}(0) = a_{20}(0)$ ,  $k_1 a_{10y}(0) = k_2 a_{20y}(0)$ . Relations (6) result from the equality of temperatures and heat fluxes on the interface  $y = 0$  [3].

Let us formulate the problem for  $w_j(y, t)$ . Due to (2)

$$b_j(y, t) = \rho_j g \beta_j \int_0^y a_j(z, t) dz + C_j(t), \quad (7)$$

with arbitrary functions  $C_j(t)$ , then

$$w_{jt} = \nu_j w_{jyy} + g \beta_j \int_0^y a_j(z, t) dz + \frac{1}{\rho_j} C_j(t), \quad (8)$$

$$w_j(y, 0) = w_{j0}(y), \quad (9)$$

$$w_1(-h_1, t) = 0, \quad w_2(h_2, t) = 0. \quad (10)$$

The dynamic condition at the interface [3] is reduced into two conditions:

$$\rho_2 \nu_2 w_{2y}(0, t) - \rho_1 \nu_1 w_{1y}(0, t) = -a_1(0, t) \alpha, \quad p_1(0, t) = p_2(0, t). \quad (11)$$

From the latter condition and the representations (1), (7) we obtain the equalities  $C_1(t) = C_2(t) \equiv C(t)$ ,  $P_1(0, t) = P_2(0, t)$ . Besides,

$$w_1(0, t) = w_2(0, t) \quad (12)$$

and the initial data for a smooth solution must be agreed:  $w_{10}(-h_1) = 0$ ,  $w_{20}(h_2) = 0$ ,  $A_{10}(0) = A_{20}(0)$ ,  $\rho_2 \nu_2 w_{20y}(0) - \rho_1 \nu_1 w_{10y}(0) = -A_{10}(0) \alpha$ ,  $w_{10}(0) = w_{20}(0)$ .

If the function  $C(t)$  is given, then the equation (8) with the boundary conditions (10) – (12) are the problem for the velocities. The aim of this paper is to study the inverse problem. Therefore, it is necessary to impose additional condition. We assume that the total fluid flow rate in layers is given

$$\int_{-h_1}^0 w_1(y, t) dy + \int_0^{h_2} w_2(y, t) dy = Q(t). \quad (13)$$

Similar formulations of problems arise under studying some technological processes, namely: thermal stabilization of power plants or cooling of electronic devices, no - contact coating technologies or alloying of steel [4, 5, 6].

## 2. Stationary flow

As before, we assume that  $A_1^s = \text{const} \neq 0$  on the bottom wall  $y = -h_1$ . Now, we give the form of the stationary solution. The following dimensionless variables and parameters are introduced:  $\xi = y/h_1$ ,  $-1 \leq \xi \leq 0$ ,  $\eta = y/h_2$ ,  $0 \leq \eta \leq 1$ ,  $k = k_1/k_2$ ,  $h = h_1/h_2$ ,  $\chi = \chi_1/\chi_2$ ,  $\rho = \rho_1/\rho_2$ ,  $\nu = \nu_1/\nu_2$ ,  $a = A_2^s/A_1^s = \text{const}$ ,

$$\text{Ra}_1 = \frac{g\beta_1 A_1^s h_1^4}{\nu_1 \chi_1}, \quad \text{Ra}_2 = \frac{g\beta_2 A_1^s h_2^4}{\nu_2 \chi_2}, \quad \bar{C}^s = \frac{h_1^3 C^s}{\rho_1 \nu_1 \chi_1}, \quad \text{Ma} = \frac{\alpha A_1^s h_2^2}{\rho_2 \nu_2 \chi_2},$$

then

$$a_1^s(\xi) = A_1^s \left[ 1 + \frac{h(a-1)(1+\xi)}{k+h} \right], \quad a_2^s(\eta) = A_1^s \left[ 1 + \frac{(a-1)(h+k\eta)}{k+h} \right], \quad (14)$$

$$w_1^s(\xi) = \frac{\chi_1}{h_1} \left\{ -\frac{\xi^2}{2} \bar{C}^s - \frac{\text{Ra}_1}{6} \left[ \xi^3 + \frac{h(a-1)}{k+h} \left( \xi^3 + \frac{\xi^4}{4} \right) \right] \right\} + \alpha_1 \xi + \alpha_2, \quad (15)$$

$$w_2^s(\eta) = \frac{\chi_2}{h_2} \left\{ -\frac{\rho\nu\chi}{2h^3} \bar{C}^s \eta^2 - \frac{\text{Ra}_2}{6} \left[ \eta^3 + \frac{(a-1)}{k+h} \left( h\eta^3 + \frac{k\eta^4}{4} \right) \right] \right\} + \alpha_3 \eta + \alpha_2,$$

$$\alpha_1 = \frac{\chi_1 h}{h_1(h+\rho\nu)} \left\{ \frac{\bar{C}^s}{2} \left( \frac{\rho\nu}{h^2} - 1 \right) + \frac{\text{Ra}_1}{6} \left[ 1 + \frac{3h}{4} \frac{(a-1)}{k+h} \right] + \right. \\ \left. + \frac{h}{6\chi} \text{Ra}_2 \left[ 1 + \frac{(a-1)}{k+h} \left( h + \frac{k}{4} \right) \right] + \frac{h}{\chi} \text{Ma} \left[ 1 + \frac{h(a-1)}{k+h} \right] \right\},$$

$$\alpha_2 = \alpha_1 + \frac{\chi_1}{h_1} \left\{ \frac{\bar{C}^s}{2} - \frac{\text{Ra}_1}{6} \left[ 1 + \frac{3h}{4} \frac{(a-1)}{k+h} \right] \right\},$$

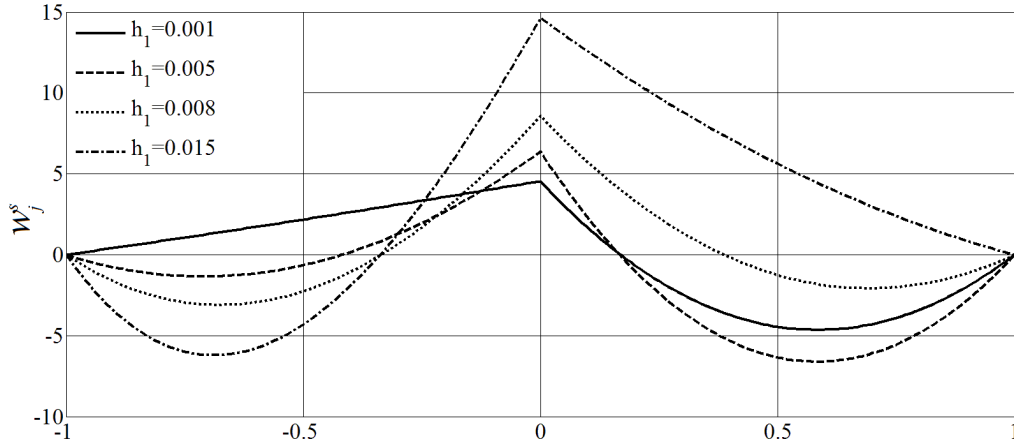
$$\alpha_3 = \frac{\chi_2 h}{h_2(h+\rho\nu)} \left\{ \frac{\rho\nu}{h} \left[ \frac{\chi \bar{C}^s}{2h} \left( \frac{\rho\nu}{h^2} - 1 \right) + \frac{\chi \text{Ra}_1}{6h} \left( 1 + \frac{3h}{4} \frac{(a-1)}{k+h} \right) + \right. \right. \\ \left. \left. + \frac{\text{Ra}_2}{6} \left( 1 + \frac{(a-1)}{k+h} \left( h + \frac{k}{4} \right) \right) \right] - \text{Ma} \left[ 1 + \frac{h(a-1)}{k+h} \right] \right\}.$$

In the considered inverse problem, with given total fluid flow rate  $Q^s = \text{const}$  (see (13)), dimensionless constant  $\bar{C}^s$  is defined as follows

$$\bar{C}^s = \frac{6h^3\chi^{-1}}{\alpha_5} \left\{ \bar{Q}^s - \chi \text{Ra}_1 \left[ \frac{1}{24} + \frac{h(a-1)}{30(k+h)} - \left( \frac{1}{6} + \frac{h(a-1)}{8(h+k)} \right) \left( 1 + \frac{1-\alpha_4}{h} \right) \right] + \right. \\ \left. + \text{Ra}_2 \left[ \frac{1}{24} + \frac{(a-1)(k+5h)}{120(k+h)} - \alpha_4 \left( \frac{1}{6} + \frac{(a-1)(k+4h)}{24(h+k)} \right) \right] + \frac{\text{Ma}(1-2\alpha_4)}{2(k+h)} \right\}, \quad (16)$$

where  $\bar{Q}^s = Q^s/\chi_2$  is dimensionless flow,  $\alpha_4 = (\rho\nu + h(h+2))(\rho\nu + h)^{-1}/2$ ,  $\alpha_5 = h^2(2h+3-3\alpha_4) + \rho\nu(1+3\alpha_4)$ . We have  $a = 1$  at equal temperature gradients on the walls  $y = -h_1$ ,  $y = h_2$  and the all formulas are simplified.

Figure 1 present the profiles of dimensionless velocities  $\bar{w}_j^s = w_j^s h_j \chi_j^{-1}$  depending on the thickness of the first layer for the system water ( $j = 1$ ) – liquid CO<sub>2</sub> ( $j = 2$ ). The physical parameters of the system are the following:  $\rho_1 = 0.99821 \cdot 10^3 \text{ kg/m}^3$ ,  $\nu_1 = 1.0038 \cdot 10^{-6} \text{ m}^2/\text{s}$ ,  $k_1 = 0.5984 \text{ kg} \cdot \text{m/s}^3 \cdot \text{K}$ ,  $\chi_1 = 1.44152 \cdot 10^{-7} \text{ m}^2/\text{s}$ ,  $\beta_1 = 0.18 \cdot 10^{-3} \text{ K}^{-1}$ ,  $\rho_2 = 0.867191 \cdot 10^3 \text{ kg/m}^3$ ,  $\nu_2 = 2.0076 \cdot 10^{-6} \text{ m}^2/\text{s}$ ,  $\chi_2 = 1.1221 \cdot 10^{-7} \text{ m}^2/\text{s}$ ,  $k_2 = 0.1101056 \text{ kg} \cdot \text{m/s}^3 \cdot \text{K}$ ,  $\beta_2 = 1.0188 \cdot 10^{-3} \text{ K}^{-1}$ ,  $\alpha = 0.0001989 \text{ N/m} \cdot \text{K}$ ,  $\sigma = 72.86 \cdot 10^{-3} \text{ N/m}$ . If the thicknesses of the layers are the same  $h_1 = h_2 = 0.001 \text{ m}$ , a zone with the reverse flow arises in the second layer and velocity profile is a parabolic one. The flow is the Poiseuille one. With increasing thickness of the first layer the reverse flow is formed only in the first layer and the velocity profile becomes linear. Such flow is the Couette flow.



**Figure 1.** Stationary profile of dimensionless velocities  $\bar{w}_j^s$ ;  $A_1^s = 2^\circ\text{C}/\text{m}$ ;  $Q^s = 0$ ;  $h_2 = 0.001\text{m}$

### 3. A priori estimates

In solving the initial conjugate initial boundary value problems (3) – (6), (8) – (13) the method of variables separation is not applicable. Therefore, for the study the qualitative properties of the solutions, we use the method of a priori estimates [7]. Using this technique we can show that the solution of initial boundary value problem (3) – (6) tends to stationary one (14) if the integrals

$$\int_0^\infty e^{\delta t} |A_j(t) - A_j^s| dt, \quad \int_0^\infty e^{\delta t} |A_j'(t)| dt, \quad \delta = \frac{1}{M} \min_j \left( \frac{k_j}{\chi_j} \right), \quad j = 1, 2, \quad (17)$$

converge and we have estimates for the convergence rate are valid

$$|a_j(y, t) - a_j^s(y)| \leq \left( \frac{8\chi_j}{k_j^2} F(t)K(t) \right)^{1/4} + |A_j(t) - A_j^s|. \quad (18)$$

The last estimates are uniform ones with respect to  $y \in [-h_1, 0]$  for  $j = 1$  and  $y \in [0, h_2]$  at  $j = 2$  and  $t \in [0, \infty)$ . In (17)  $M$  is the constant from the Friedrichs inequality [8]; the function  $F(t)$  in (18) is limited and  $K(t) \sim e^{-2\delta t}$ .

For the problem (8) – (13) we get similar result. The solution of initial boundary value problem (8) – (13) tends to stationary one (15), (16) if the integrals (17) and

$$\int_0^\infty e^{\delta_1 t} |A_j''(t)| dt, \quad \int_0^\infty e^{\delta_1 t} |Q(t) - Q^s| dt, \quad \int_0^\infty e^{\delta_1 t} |Q'(t)| dt, \quad \int_0^\infty e^{\delta_1 t} |Q''(t)| dt, \quad \delta_1 = \frac{1}{M} \min_j \left( \frac{1}{\rho_j} \right) \quad (19)$$

converge and we have estimates for the convergence rate are valid

$$|w_j(y, t) - w_j^s(y)| \leq \begin{cases} \frac{\sqrt{3}}{18} \frac{\alpha h_1}{\rho_1 \nu_1} |a_1(0, t) - a_1^s(0)| \\ \frac{16h_2}{9} |Q(t) - Q^s| \end{cases} + \left( \frac{8\nu_j}{\mu_j^2} F_2(t)E(t) \right)^{1/4}. \quad (20)$$

Estimates (20) are the uniform ones with respect to  $y \in [-h_1, 0]$  at  $j = 1$  and  $y \in [0, h_2]$  at  $j = 2$ ;  $t \in [0, \infty)$ . And for the function  $C(t)$  we have

$$|C(t) - C^s| \leq c \left[ |A_1(t) - A_1^s| + \left( \frac{8\chi_1}{k_1^2} F(t)K(t) \right)^{1/4} + |A_1'(t)| + \left( \frac{8\chi_1}{k_1^2} F_1(t)K_1(t) \right)^{1/4} + \right. \\ \left. + \left( \frac{8\nu_1}{\mu_1^2} F_2(t)E(t) \right)^{1/4} + \left( \frac{8\nu_1}{\mu_1^2} F_3(t)E_1(t) \right)^{1/4} \right] \quad (21)$$

with positive constant  $c$  depending on the input data. In (20), (21) the functions  $F_1(t)$ ,  $F_2(t)$ ,  $F_3(t)$  are limited and  $E(t), E_1(t) \sim e^{-2\delta_1 t}$ .

#### 4. The solution of conjugate problem by the Laplace transformation method.

##### Analysis of the numerical results

For more accurate information on behavior  $a(y, t)$  and  $w(y, t)$  the Laplace transformation is applied. As a result, we get to the boundary value problem for images  $\hat{a}_j(y, p)$  of functions  $a_j(y, p)$

$$\hat{a}_{1\xi\xi}(\xi, p) - p\hat{a}_1(\xi, p) = -a_{10}(\xi), \quad (22)$$

$$\hat{a}_{2\eta\eta}(\eta, p) - p\chi h^{-2}\hat{a}_2(\eta, p) = -\chi h^{-2}a_{20}(\eta),$$

$$\hat{a}_1(-1, p) = \hat{A}_1(p), \quad \hat{a}_2(1, p) = \hat{A}_2(p), \quad (23)$$

$$\hat{a}_1(0, p) = \hat{a}_2(0, p), \quad kh^{-1}\hat{a}_{1\xi}(0, p) = \hat{a}_{2\eta}(0, p)$$

and images  $\hat{w}_j(y, t)$  of functions  $w_j(y, t)$

$$\hat{w}_{1\xi\xi}(\xi, p) - p\text{Pr}_1^{-1}\hat{w}_1(\xi, p) = - \left( \text{Pr}_1^{-1}w_{10}(\xi) + Ra_1 \int_0^\xi \hat{a}_1(z, p) dz + \hat{C}(p) \right), \quad (24)$$

$$\hat{w}_{2\eta\eta}(\eta, p) - \frac{p\chi}{h^2\text{Pr}_2}\hat{w}_2(\eta, p) = - \left( \frac{\chi}{h^2\text{Pr}_2}w_{20}(\xi) + Ra_2 \int_0^\eta \hat{a}_2(z, p) dz + \frac{\rho\nu\chi}{h^3}\hat{C}(p) \right),$$

$$\hat{w}_1(-1, p) = \hat{w}_2(1, p) = 0, \quad \chi h^{-1}\hat{w}_1(0, p) = \hat{w}_2(0, p),$$

$$\hat{w}_{2\eta}(0, p) - \rho\nu\chi h^{-2}\hat{w}_{1\xi}(0, p) = -Ma\hat{a}_1(0, p), \quad \chi \int_{-1}^0 \hat{w}_2(\xi, p) d\xi + \int_0^1 \hat{w}_2(\eta, p) d\eta = \hat{Q}(p). \quad (25)$$

The problems (22), (23) and (24), (25) are written in a dimensionless form with the characteristic scales  $h_1^2/\chi_1^{-1}$ ,  $A_1^s$ ,  $\chi_j/h_j^{-1}$  for time and functions  $a_j(y, t), w_j(y, t)$ , respectively. Upon that  $A_1^s = \max_t |A_1(t)| > 0$ ,  $\text{Pr}_j = \nu_j/\chi_j$  are the Prandtl numbers. In deriving equations (22), (23) the initial data (4), (9) are used. In (23), (25)  $\hat{A}_1(p)$ ,  $\hat{A}_2(p)$  and  $\hat{Q}(p)$  are images of given functions  $A_1(t)$ ,  $A_2(t)$  and  $Q(t)$ , respectively (see conditions (5) and (13)).

The general solution of equations (22) has the form

$$\hat{a}_1(\xi, p) = m_1 \text{sh}\sqrt{p}\xi + m_2 \text{ch}\sqrt{p}\xi - \frac{1}{\sqrt{p}} \int_{-1}^\xi a_{10}(z) \text{sh}\sqrt{p}(\xi - z) dz, \quad (26)$$

$$\hat{a}_2(\eta, p) = m_3 \text{sh}\sqrt{p\chi}h^{-1}\eta + m_4 \text{ch}\sqrt{p\chi}h^{-1}\eta - \frac{1}{h\sqrt{p\chi^{-1}}} \int_0^\eta a_{20}(z) \text{sh}\sqrt{p\chi}h^{-1}(\eta - z) dz.$$

Similarly, the solution of equations (24) has the form

$$\begin{aligned}
 \hat{w}_1(\xi, p) &= C_1 \text{sh} \alpha_1 \xi + C_2 \text{ch} \alpha_1 \xi - \frac{1}{\alpha_1} \int_0^\xi f_1(z, p) \text{sh} \alpha_1 (\xi - z) dz - \frac{\hat{C}(p)}{\alpha_1} (\text{ch} \alpha_1 \xi - 1), \\
 \hat{w}_2(\eta, p) &= C_3 \text{sh} \alpha_2 \eta + C_4 \text{ch} \alpha_2 \eta - \frac{1}{\alpha_2} \int_0^\eta f_2(z, p) \text{sh} \alpha_2 (\eta - z) dz - \frac{\rho \nu \text{Pr}_2 \hat{C}(p)}{p h} (\text{ch} \alpha_2 \eta - 1), \\
 f_1(\xi, p) &= \text{Pr}_1^{-1} w_{10}(\xi) + \text{Ra}_1 \int_0^\xi \hat{a}_1(z, p) dz, \quad f_2(\eta, p) = \frac{\chi}{h^2 \text{Pr}_2} w_{20}(\xi) + \text{Ra}_2 \int_0^\eta \hat{a}_2(z, p) dz, \\
 \alpha_1 &= \sqrt{p \text{Pr}_1^{-1}}, \quad \alpha_2 = \sqrt{p \chi \text{Pr}_2^{-1} h^{-1}}.
 \end{aligned} \tag{27}$$

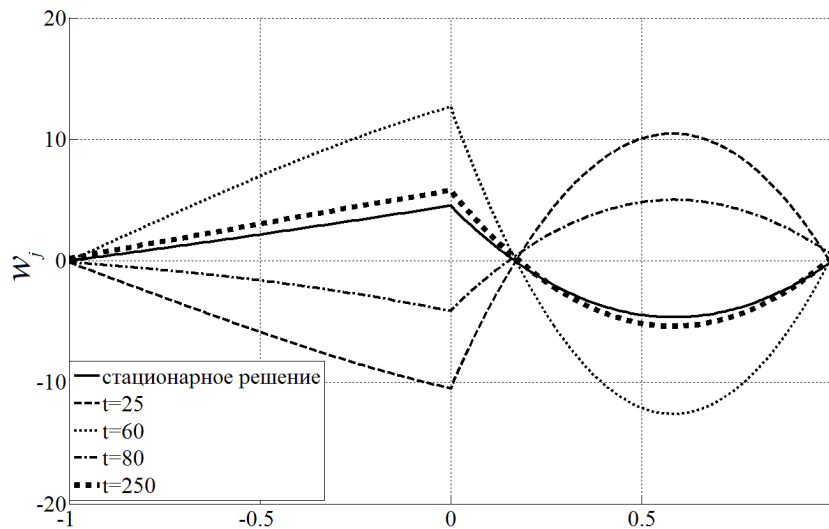
The values  $m_k$ ,  $C_k$ ,  $k = 1, \dots, 4$ , including in (26), (27), and the function  $\hat{C}$  are determined from boundary conditions (23), (25) have complex expressions and are not given here.

Suppose that  $\lim_{t \rightarrow \infty} A_j(t) = A_j^s$  and  $\lim_{t \rightarrow \infty} Q(t) = Q^s$ . Using the obtained representations for  $\hat{a}_j(y, p)$ ,  $\hat{w}_j(y, p)$  and  $\hat{C}(p)$  we can prove the limit equalities

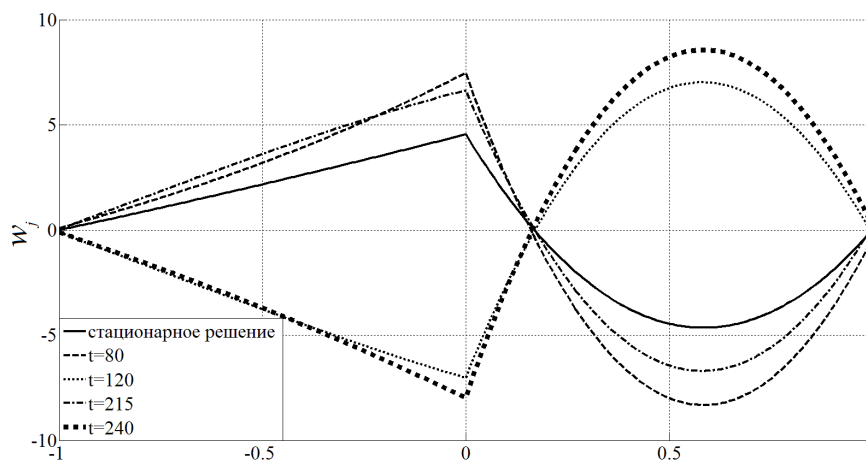
$$\begin{aligned}
 \lim_{t \rightarrow \infty} a_j(y, t) &= \lim_{p \rightarrow 0} p \hat{a}_j(y, p) = a_j^s(y), \quad \lim_{t \rightarrow \infty} w_j(y, t) = \lim_{p \rightarrow 0} p \hat{w}_j(y, p) = w_j^s(y), \\
 \lim_{t \rightarrow \infty} C(t) &= \lim_{p \rightarrow 0} p \hat{C}(p) = C^s,
 \end{aligned} \tag{28}$$

where  $a_j^s(y)$ ,  $w_j^s(y)$  and  $C^s$  are given by formulas (14) – (16). It is in agreement with the conclusions of the Section 3.

Let us apply the method of the numerical inversion of the Laplace transformation to obtain some results for the velocity functions. Let  $Q(t) = Q^s = 0$  (the motion is caused by the action of the thermogravitational forces) and the longitudinal temperature gradient on the walls is distributed according to the law  $A_j(t) = \gamma_j^1 + \gamma_j^2 e^{-\gamma_j^3 t} \sin(\gamma_j^4 t)$ ,  $j = 1, 2$ , where the coefficients  $\gamma_j^2, \gamma_j^4$  correspond to the amplitude and frequency of the oscillations, respectively. If  $\gamma_j^1 \neq 0, \gamma_j^3 > 0$  then takes place convergence of the solution to the stationary regime, according to equations (28). If  $\gamma_j^3 \leq 0$ , then limits of the function  $A_j(t)$  at  $t \rightarrow \infty$  do not exist then the solution does not tend to a stationary regime. Figure 2 and Figure 3 correspond to cases  $A_1(t) = 1 - 5e^{-0.01t} \sin(0.1t)$  and  $A_1(t) = 2 \sin(0.1t)$ , respectively. As expected, the solution does not tend to a stationary regime with increasing time in the latter example.



**Figure 2.** Dimensionless velocities profile  $w_j(y, t)$  at  $A_1(t) = 1 - 5e^{-0.01t} \sin(0.1t)$ ,  $A_2(t) = 0$ ,  $h_1 = h_2 = 0.001\text{m}$



**Figure 3.** Dimensionless velocities profile  $w_j(y, t)$  at  $A_1(t) = 2 \sin(0.1t)$ ,  $A_2(t) = 0$ ,  $h_1 = h_2 = 0.001\text{m}$

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